Lec 19 : Signed measures, Hahn-Jordan decomposition

Linear combinations: μ_1, μ_2 measures on E and $\alpha\mu_1 + \beta\mu_2$ is also a meas for $\alpha, \beta > 0$.

1) $\alpha\mu_1 + \beta\mu_2(\phi) = 0$ 2) countable additivity (easy)

Would like to define measures with not just nonnegative coeffients but where α or β are allowed to be negative.

Signed meas: Space $(X, \mathcal{F})$ $\nu : \mathcal{F} \to [-\infty, \infty]$

1) ν takes either $+\infty$ or $-\infty$ but not both
2) $\nu(\phi) = 0$
3) Countable additivity.

If $\{E_k\}$ disjoint and $\nu(\bigcup_{k=1}^{\infty} E_k)$ finite, then

$$\nu\left(\bigcup_k E_k\right) = \sum_{k=1}^{\infty} \nu(E_k)$$ and the series

converges absolutely.

Why do we care? Suppose f is integrable on $(X, \mathcal{F}, m)$ for any set E we will define

$$\nu(E) = \int_E f \, dm$$

Then ν is a signed measure and it has the following Hahn-Jordan decomposition.

Let $A = \{x \in X \mid f \geq 0\}$ and $B = \{x \in X \mid f < 0\}$

Then $\nu(E) = \nu^+(E) - \nu^-(E)$

where $\nu^+(E) = \int_{A \cap E} f$ $\nu^-(E) = \int_{B \cap E} -f$

ν^+ and ν^- are <u>measures</u> (unsigned).

<u>Positive sets</u> A set E is positive if $\forall A \subset E$ and $A \in \mathcal{F}$,

$\nu(A) \geq 0$?.

<u>Negative sets</u> Similar

<u>NULL Sets</u> E meas and $\forall A \subset E$ $\nu(E) = 0$ A?.

If $A \subseteq B$ and $|\nu(B)| < \infty$ we must have by countable additivity that

$$\nu(B) = \nu(A) + \nu(B \setminus A)$$

Then if $\nu(A) = +\infty$, $\nu(B \setminus A) \neq -\infty$ since signed measures only take $+\infty$ or $-\infty$. Similarly $\nu(A)$ cannot be $-\infty$ and so

$$|\nu(A)| < \infty \quad \text{if} \quad |\nu(B)| < \infty.$$

<u>Prop</u>: If E disjoint union $E = \bigcup_n E_n$ of positive sets then E is positive.

Pf: Take any $A \subset E$, and break it up into a disjoint union $\bigcup_n (A \cap E_n)$. Each of them has positive meas.

We're done by countable additivity.

<u>Hahn's lemma</u>: If $0 < \nu(E) < \infty$ then $\exists$ a subset of E that is positive and has pos. measure.

Pf: We simply remove all the negative sets from E.

Let $m_1 = \inf\{n \in \mathbb{N} \mid \exists A \subset E, \nu(A) < -\frac{1}{n}\}$ Let $A_1 = A$

Inductively $m_k = \{n \in \mathbb{N} \mid \exists A \subset E, A \cap \underbrace{\bigcup_{r \geq 1}^{k-1} E_r}_{\text{bad sets}} = \phi, \nu(A) < -\frac{1}{n}\}$

If at any point $m_n = +\infty$ there is no n st $\nu(A) < -\frac{1}{n}$ and $A \cap \bigcup_{r=1}^{n-1} E_r = \phi$ and $A \subset E$

This means every subset of $E \setminus \bigcup_{r=1}^{n-1} E_r$ MUST have nonnegative measure. Thus $E \setminus \bigcup_{r=1}^{n-1} E_r$ is the positive set we desire. So assume we can find a sequence of such m_n. Could these m_n run of to $+\infty$ or must they stay bounded.

☆ Could we have chosen $m_k > m_{k-1}$? I don't see why not at the moment. Worth thinking about.

In any case since $|\nu(E)| < \infty$ and $\bigcup_{k=1}^{\infty} E_k \subset E$ we must have $\left|\nu\left(\bigcup_{k=1}^{\infty} E_k\right)\right| < \infty$. Therefore,

$$-\infty < \nu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} -\frac{1}{m_k} \leq 0$$

$\Rightarrow m_k \longrightarrow \infty$.

Now we claim that $E \setminus \bigcup_{k=1}^{\infty} E_k$ is the set of positive meas.
Take any $B \subset E \setminus \bigcup_{k=1}^{\infty} E_k$. Choose n st $m_n > 1$. Given $B \subset E \setminus \bigcup_{k=1}^{n} E_k$ for any n, $\nu(B) > -\frac{1}{m_n - 1}$; for if not m_n is not the smallest integer k such that $\nu(B) \leq -\frac{1}{k}$, it would be $m_n - 1$ instead.

Thus $\nu(B) > -\frac{1}{m_n - 1}$ $\forall$ n and hence $\nu(B) \geq 0$. This proves positivity.

To show that $\nu(E \setminus \bigcup_{k=1}^{\infty} E_k) > 0$, note that

$$\nu(E \setminus \bigcup_{k=1}^{\infty} E_k) = \nu(E) - \sum_{k=1}^{\infty} \nu(E_k) \geq \nu(E) + \sum_{k=1}^{\infty} \frac{1}{m_k}$$

Since $\nu(E) > 0$ we're done.

<u>Hahn decomposition Theorem</u>: Let $(X, \mathcal{F})$ and ν be a signed measure. Then there are A, B st

$$X = A \cup B \quad \nu(A) \geq 0, \quad \nu(B) \leq 0 \quad \text{and}$$

and $A \cap B = \phi$

Pf: Assume ν does not take $+\infty$

Let $\mathcal{P} = \{ E \subset X \mid E \text{ positive} \}$ and

$$\lambda = \sup_{A \in \mathcal{P}} \nu(A)$$

Let $\{A_n\}$ st $\lambda = \lim_{k \to \infty} \nu(A_n)$ and $A = \bigcup_{k=1}^{\infty} A_n$.

A is positive, so $A \setminus A_n \subseteq A$ and $\nu(A \setminus A_n) \geq 0$.
Then

$$\nu(A) = \nu(A \setminus A_n) + \nu(A_n) \geq \nu(A_n)$$

and hence $\nu(A) \geq \lambda \quad (k \to \infty)$.

Let $B = X \setminus A$. We argue that B is negative.

If B is not negative, then $\exists$ a $B' \subset B$ st $\nu(B') > 0$

By Hahn's Lemma $\exists$ $B'' \subset B'$ st B'' positive. Then

$$\nu(A \cup B'') = \nu(A) + \nu(B'') > \lambda \quad \text{and this is a}$$

contradiction.

Such a decomposition is not unique.

If $\{A, B\}$ is a Hahn decomposition of X, it is not generally unique since you can add null sets to A or B and they would each remain positive or negative resp.

Mutualy singular measures: If ν_1, ν_2 measures on X they're mutually singular if $\exists\, A, B$ st

$X = A \cup B \quad A \cap B = \phi \quad$ and $\nu_1(B) = 0$ and $\nu_2(A) = 0$

We write $\nu_1 \perp \nu_2$

Jordan's Theorem: Let $(X, \mathcal{F})$ and ν be a signed meas. space. $\exists!\, \nu^+$ and ν^- such that

$$\nu = \nu^+ - \nu^- \quad \text{and } \nu^+ \perp \nu^-$$

Moreover ν^+ and ν^- are unique.

Pf: Obtain A, B from Hahn and define

$$\nu^+(E) = \nu(A \cap E) \quad (\geq 0)$$

$$\nu^-(E) = -\nu(B \cap E)$$

Then by additivity $\nu(E) = \nu^+(E) - \nu^-(E)$

17.3 Caratheodory Measure induced by outer measure

Define: $\mu : \Sigma \to [0,\infty]$ defined on a collection Σ of subsets of a set X is countably (subadditive) monotone if

$$E \subset \Sigma \text{ , and } \exists \ \{E_k\}_{k=1}^{\infty} \text{ such that}$$

E is covered by $E \subset \bigcup_{k=1}^{\infty} E_k$. Then

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

If $\phi \in \Sigma$ and $\mu(\phi) = 0$ then μ is also FINITELY MONOTONE on Σ.

Finite Monotonicity $\Rightarrow$ Monotonicity

since if $A \subset B \in \Sigma$ then B is a cover of A

and hence $\mu(A) \leq \mu(B)$.

General Outer Measure: $\mu^* : 2^X \to [0,\infty]$ is called an outer measure provided $\mu^*(\phi) = 0$ and μ^* is countably subadditive/monotone

Definition: $\mu^* : 2^X \to [0,\infty]$ outer measure. We call E measurable if it satisfies the Caratheodory Criterion:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Same route as Lebesgue measure.

It's worth commenting on this criterion: why this particular form?

How did Carathéodory discovery.

The only difference between $\underset{\substack{\uparrow \\ \text{outer}}}{\mu^*}$ and $\underset{\substack{\uparrow \\ \text{measure}}}{\mu}$ is that measures are

countably <u>additive</u>.

μ^* is defined on 2^X, so we want to restrict μ^* to a
a sub collection where μ^* is countably additive.

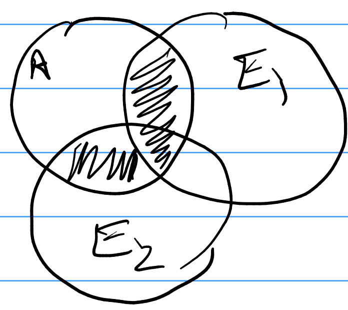
The key is prop. 6.

$$\mu^*\left(A \cap \bigsqcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k)$$

<u>Prop 5</u>: Union of a finite collection of sets is measurable.

Pf: $E = E_1 \cup \cdots \cup E_n$. Do it for $n=2$ case.

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)$$
$$= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c)$$
$$= \mu(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c)$$



For the general inductive step, set $\tilde{E}_1 = \bigcup_{k=1}^{n-1} E_k$, $\tilde{E}_2 = E_n$ and apply the same argument.

Prop 6 Let $A \subseteq X$ and $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then

$$\mu\left(A \cap \bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu(A \cap E_k)$$

Pf: Again it's a proof by induction on n.

$$A \cap \bigcup_{k=1}^n E_k \cap E_n = A \cap E_n \quad \text{(disjointness)}$$

$$A \cap \bigcup_{k=1}^n E_k \cap E_n^c = A \cap \bigcup_{k=1}^{n-1} E_k$$

$$\mu\left(A \cap \bigcup_{k=1}^n E_k\right) = \mu(A \cap E_n) + \mu\left(A \cap \bigcup_{k=1}^{n-1} E_k\right)$$

using the measurability of E_n

Prop 7 The union of a countable collection of measurable sets is measurable

Pf: We may assume $\{E_k\}_{k=1}^\infty$ are disjoint and measurable.

To show for any $A \subset X$, we have $\mu(A) \geq \mu(A \cap \bigcup_k E_k) + \mu(A \cap [\bigcup_k E_k]^c)$

Fix $A \subset X$. $F_n = \bigcup_{k=1}^{n} E_k$

$$\mu(A) = \mu(A \cap F_n) + \mu(A \cap F_n^c)$$
$$\geq \mu(A \cap F_n) + \mu(A \cap [\bigcup_k E_k]^c)$$
$$= \sum_{k=1}^{n} \mu(A \cap F_n) + \mu(A \cap [\bigcup_k E_k]^c)$$

Take a limit as $n \to \infty$ and then we countable monotonicity to get

$$\geq \sum_{k=1}^{\infty} \mu(A \cap \bigcup_k E_k) + \mu(A \cap (\bigcup_k E_k)^c)$$

<u>Theorem</u>: Let μ^* be an outer meas on 2^X. Then $\mathcal{F}$ the collection of measurable sets is a σ-algebra.

If $\bar{\mu} = \mu^*|_{\mathcal{F}}$ then $(X, \mathcal{F}, \bar{\mu})$ is a COMPLETE meas space.

<u>Pf</u>: ($\mathcal{F}$ is closed under complements and countable unions. Since $\mu^*(\phi) = 0$ we know it satisfies

$$\mu^*(A) = \mu^*(A \cap \phi) + \mu^*(A \cap \phi^c).$$ So ϕ is meas.

As before we only need to show that

$$\mu^*(\bigcup_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} \mu(E_k) \qquad E_k \text{ disjoint.}$$

We have showed previously that for any meas. A that

$\mu^*(A \cap \bigcup_n E_n) \geq \sum_n \mu(E_n \cap A)$ — simply set $A = X$.

How about completeness? You will prove that if $\mu^*(E) = 0$, E is meas.

Let $E \subset B$, B measurable $\mu^*(B) = 0 \Rightarrow \mu^*(E) = 0$

$\mu^*(A) \geq \mu^*(A \cap E^c)$ (monotonicity)

But $\mu^*(A \cap E) \leq 0$ (by monotonicity

$\Rightarrow \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$